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Hölder exponents of irregular signals and local fractional derivatives

Kiran M. Kolwankar[1] and Anil D. Gangal[2]

Department of Physics, University of Pune, Pune 411 007, India.

Abstract

It has been recognized recently that fractional calculus is useful for handling scaling structures and processes. We begin this survey by pointing out the relevance of the subject to physical situations. Then the essential definitions and formulae from fractional calculus are summarized and their immediate use in the study of scaling in physical systems is given. This is followed by a brief summary of classical results. The main theme of the review rests on the notion of local fractional derivatives. There is a direct connection between local fractional differentiability properties and the dimensions/ local Hölder exponents of nowhere differentiable functions. It is argued that local fractional derivatives provide a powerful tool to analyse the pointwise behaviour of irregular signals and functions.

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1 Introduction

In the 1880's, contrary to the notion existing till then that a continuous function must be differentiable *at least at some point*, examples were constructed [1] to demonstrate that it is possible for continuous functions to be *no where* differentiable. Weierstrass's construction provided one such early example. For about three decades after the construction of such functions, they were still considered to be rather pathological cases without any practical relevance to the physical world. Perrin was the first to point out their application in a real physical situation, viz., the problem of Brownian motion. He also prophesied [3] the possibility of a simpler way of handling these functions than differentiable ones.

Recent developments in nonlinear and nonequilibrium phenomena suggest that such irregular functions occur much more naturally and abundantly in formulations of physical theories. The work on Brownian motion (see [4]) showed that the graphs of projections of Brownian paths are nowhere differentiable and have dimension 3/2. A generalization of Brownian motion, called fractional Brownian motion, [3, 4], is known to give rise to graphs having dimension between 1 and 2. It was also observed that typical Feynmann paths [5, 6], like the Brownian paths, are continuous but nowhere differentiable. In fluid systems, passive scalars advected by a turbulent fluid have been shown [7, 8] to have isoscalar surfaces which are highly irregular, in the limit of diffusion constant going to zero. Also there exist situations in which one has to solve a partial differential equation subject to fractal boundary conditions, such as Laplace equation near a fractal conducting surface. For instance, as noted in reference [9] irregular boundaries may appear to be non-differentiable

[1]email: kirkol@physics.unipune.ernet.in
[2]email: adg@physics.unipune.ernet.in

everywhere upto a certain spatial resolution, and/or may exhibit convolutions over many length scales. Nowhere differentiable functions can be used to model these boundaries.

In the study of dynamical systems theory, attractors of some systems are found to be continuous but nowhere differentiable [10]. We will consider one specific example [10, 11] of the dynamical system which gives rise to such an attracter. Consider the following map.

$$\begin{aligned} x_{n+1} &= 2x_n + y_n \quad \mod 1, \\ y_{n+1} &= x_n + y_n \quad \mod 1, \\ z_{n+1} &= \lambda z_n + \cos(2\pi x_n) \end{aligned} \tag{1}$$

where x and y are taken mod 1 and z can be any real number. In order to keep z bounded, λ is chosen between 0 and 1. The equations for x and y are independent of z. The $x - y$ dynamics are chaotic and are unaffected by the value of z. It was shown in [10] that if $\lambda > 2/(3 + \sqrt{5})$, the attracter of this map is nowhere differentiable torus. One more example in 2-dim phase space is given in section 5.

All these irregular functions are best characterized locally by a *Hölder exponent*. We will use following general definition of the Hölder exponent $h(y)$ which has been used by various authors [12, 13] recently. The exponent $h(y)$ of a function f at y is given by h such that there exists a polynomial $P_n(x)$ of order n, where n is the largest integer smaller than h, that satisfies

$$|f(x) - P_n(x - y)| = O(|x - y|^h), \tag{2}$$

for x in the neighbourhood of y. This definition serves to classify the behavior of the function at y.

Multifractal measures have been object of many investigations [14, 15, 16, 17, 18, 19, 20, 21]. This formalism has met with many applications. Its importance also stems from the fact that such measures are natural measures to be used in the analysis of many phenomenon [22, 23]. The first mathematical rigorous results concerning multifractals were given in [24]. It may however happen that the object one wants to understand is a function (e.g. a fractal or multifractal signal) rather than a set or a measure. For instance one would like to characterize the velocity field of fully developed turbulence. Studies on fluid turbulence have shown existence of multifractality in velocity fields of a turbulent fluid at low viscosity [14].

In the case of functions having same Hölder exponent h, with $0 < h < 1$, at every point (Weierstrass function (section 5) is such a example) it is well known [4] that the box dimension of its graph is $2 - h$. On the other hand, there are functions $f(x)$ which do not have constant exponent h at every point but have a range of such exponents. The set of points having same exponents $\{x \mid h(x) = h\}$ for such a function may even constitute a fractal set. In such situations corresponding functions $f(x)$ are multifractal.

In the light of these findings there has been a renewed interest, chiefly among mathematicians, in studying pointwise behaviour of multifractal functions. New tools have been developed to study local behaviour of functions, which will be of practical use. Notable among them is the use of Wavelet transforms [25, 26, 27, 28] which have been used with some success to this effect. This transform allows one to study the behaviour of functions at different scales and hence it is popularly known as a mathematical microscope. Some well-known functions have been reanalyzed using these techniques [29, 30, 31, 32] and shown to possess a spectrum of Hölder exponents, thereby establishing their multifractality.

In this article we review a more direct method, which uses fractional calculus formalism, to characterize local behaviour of functions. The fractional calculus formalism [33, 34, 35] allows taking derivatives of fractional order. However, as pointed out in section 2 below, there is a multiplicity of definitions of fractional derivatives. Hence it is important to recognize the appropriate one which is suitable for extracting the scaling behaviour. Recently it was realized [36] that the concept of local fractional derivative, introduced in section 4 below in a more general setting, is ideally suited for this purpose and that there is a direct quantitative connection between the lack of differentiability of a fractal function and the dimension of its graph. Further it provides an alternative way of characterizing the local Hölder exponent. In this formalism, unlike in wavelet transform, Hölder exponent shows up naturally since the magnitude of *local fractional derivative* (LFD) jumps when its order crosses a *critical order*. The main theme of this review is centered around this developement.

The organization of the paper is as follows. In section II we present a brief review of fractional calculus formalism with its recent applications. The next section discusses fractional differentiability using Weyl's definition and certain classical results. In the fourth section we present our definition of LFD in more general setting and its connection to generalized Taylor series. In section V we apply this definition to a specific example, viz, Weierstrass' nowhere differentiable function. It is shown that this function, at every point, is *locally fractionally differentiable* for all orders below $2-s$, but not so for orders between $2-s$ and 1, where s, $1<s<2$, is the box dimension of the graph of the function. In the next section a general result has been proved, showing relation between local fractional differentiability and local Hölder exponent. In section V we demonstrate use of LFD in unmasking isolated singularities and in the study of pointwise behavior of multifractal functions. Section VI concludes the article, pointing out a few possible consequencs of our results.

2 Fractional calculus and scaling phenomenon

Fractional calculus [33, 34, 35] is a study which deals with generalization of differentiation and integration to fractional orders. There are number of ways (not neccesarily equivalent) of defining fractional derivatives and integrations. We list here some of them, which will be used in this work. We begin by recalling the Riemann-Liouville definition of fractional integral of a real function, which is given by [33, 35]

$$\frac{d^qf(x)}{[d(x-a)]^q} = \frac{1}{\Gamma(-q)}\int_a^x \frac{f(y)}{(x-y)^{q+1}}dy \quad \text{for} \quad q<0, \tag{3}$$

where the lower limit a is some real number and of the fractional derivative

$$\frac{d^qf(x)}{[d(x-a)]^q} = \frac{1}{\Gamma(n-q)}\frac{d^n}{dx^n}\int_a^x \frac{f(y)}{(x-y)^{q-n+1}}dy \quad \text{for} \quad n-1<q<n. \tag{4}$$

Fractional derivative of a simple function $f(x)=x^p \quad p>-1$ is given by [33, 35]

$$\frac{d^qx^p}{dx^q} = \frac{\Gamma(p+1)}{\Gamma(p-q+1)}x^{p-q} \quad \text{for} \quad p>-1. \tag{5}$$

For future use we note the following properties of fractional derivatives. For $0<q<1$,

$$\frac{d^q sin(x)}{dx^q} = \frac{d^{q-1}cos(x)}{dx^{q-1}} \tag{6}$$

and

$$\frac{d^q cos(x)}{dx^q} = \frac{x^{-q}}{\Gamma(1-q)} - \frac{d^{q-1} sin(x)}{dx^{q-1}}. \tag{7}$$

Further the fractional derivative has the property (see ref [33]), viz,

$$\frac{d^q f(\beta x)}{dx^q} = \beta^q \frac{d^q f(\beta x)}{d(\beta x)^q} \tag{8}$$

which makes it suitable for the study of scaling. Note the nonlocal character of the fractional derivative and integral in the definitions equations (3) and (4). Also it is clear from equation (5) that unlike in the case of integer derivatives the fractional derivative of a constant is not zero in general.

Since the definition of fractional derivative contains a lower limit 'a', it is natural that different definitions of a fractional differentiability would arise depending on the lower limit one chooses. Weyl defined fractional derivatives, with the arbitrary limit a in (4) going to $-\infty$, as follows.

$$D^q f(x) = \frac{1}{\Gamma(n-q)} \frac{d^n}{dx^n} \int_{-\infty}^{x} \frac{f(y)}{(x-y)^{q-n+1}} dy \quad \text{for} \quad n-1 < q < n. \tag{9}$$

The merit of this definition is that the fractional derivative of a periodic function according to this definition is a periodic function, which is why it is suitable in harmonic analysis.

2.1 Applications to the study scaling in physical systems

Recent work shows that the fractional calculus formalism is useful in dealing with scaling phenomena. The purpose of this subsection is to point out a few relevant parallel developments even though they are not the part of the central theme of this paper.

Hilfer [37, 38, 39] has generalized the Eherenfest's classification of phase transition using the fractional derivative. Some recent papers [40, 41, 42, 43] indicate a connection between fractional calculus and fractal structure [3, 4] or fractal processes [44, 45, 46]. W. G. Glöckle and T. F. Nonnenmacher [45] have formulated fractional differential equations for some relaxation processes which are essentially fractal time [46] processes.

Fractional Brownian Motion

Mandelbrot and Van Ness [44] have used fractional integrals to formulate fractal processes such as fractional Brownian motion. The fractional Brownian motion is described by the probability distribution $B_H(t)$ defined by,

$$B_H(t) - B_H(0) = \frac{1}{\Gamma(H+1)} \int_{-\infty}^{t} K(t-t') dB(t'), \quad 0 < H < 1, \tag{10}$$

where $B(t)$ is an ordinary Gaussian random process with average zero and unit variance and $K(t-t')$

$$K(t-t') = \begin{cases} (t-t')^{H-1/2} & 0 \le t' < t \\ \{(t-t')^{H-1/2} - (-t)^{H-1/2}\} & t' < 0. \end{cases} \tag{11}$$

The use of the fractional integral kernel is apparent. Recently Sebastian [47] has given a path integral representation for fractional Brownian motion whose measure has fractional derivatives of paths in it.

Fractional equations for a class of Lévy type probability densities

Lévy flights [48] and Lévy walks [49, 50] have found applications in various branches of physics [51], for example in fluid dynamics [52] and polymers [53]. In [40] T. F. Nonnenmacher considered a class of normalized one sided Lévy type probability densities, which provides the length of a jump of a random walker, given by

$$f(x) = \frac{a^{\mu}}{\Gamma(\mu)} x^{-\mu-1} \exp(-a/x) \quad a > 0,\ x > 0. \tag{12}$$

It is clear that for large x

$$f(x) \sim x^{-\mu-1} \quad \mu > 0,\ x > 0. \tag{13}$$

where μ is the Lévy index. In [40] it was shown that these Lévy-type probability densities satisfy a fractional integral equation

$$x^{2q} f(x) = a^q \frac{d^{-q} f(x)}{dx^{-q}} \tag{14}$$

or equivalently a fractional differential equation given by

$$\frac{d^{n-\mu} f(x)}{dx^{n-\mu}} = a^{-\mu} \frac{d^n}{dx^n}(x^{2\mu} f(x)). \tag{15}$$

Here, $n = 1$ if $0 < \mu < 1$ and $n = 2$ if $1 < \mu < 2$, etc. An interesting observation of this work is that the Lévy index is related to the order of the fractional integral or differential operator.

Fractional equations for physical processes

Modifications of equations governing physical processes such as diffusion equation, wave equation and Fokker-Plank equation have been suggested [41, 42, 54, 55, 56] which incorporate fractional derivatives with respect to time. In refs. [41, 42] a fractional diffusion equation has been proposed for the diffusion on fractals. Asymptotic solution of this equation coincides with the result obtained numerically. Fogedby [57] has considered a generalization of the Fokker-Plank equation involving addition of a fractional gradient operator (defined as the Fourier transform of $-k^{\mu}$) to the usual Fokker-Plank equation and performed a renormalization group analysis.

Transport in chaotic Hamiltonian systems

Recently Zaslavasky [58] showed that the Hamiltonian chaotic dynamics of particles can be described by a fractional generalization of the Fokker-Plank-Kolmogorov equation which is defined by two fractional critical exponents (α, β) responsible for the space and time derivatives of the distribution function correspondingly. With certain assumptions, the following equation was derived for the transition probabilities $P(x,t)$:

$$\frac{\partial^{\beta} P(x,t)}{\partial t^{\beta}} = \frac{\partial^{\alpha}}{\partial(-x)^{\alpha}}(A(x)P(x,t)) + \frac{1}{2}\frac{\partial^{2\alpha}}{\partial(-x)^{2\alpha}}(B(x)P(x,t)) \tag{16}$$

where

$$A(x) = \lim_{\Delta t \to 0} \frac{A'(x; \Delta t)}{(\Delta t)^\beta} \tag{17}$$

and

$$B(x) = \lim_{\Delta t \to 0} \frac{B'(x; \Delta t)}{(\Delta t)^\beta} \tag{18}$$

with A' and B' being α^{th} and $2\alpha^{th}$ moments respectively. The exponents α and β have been related to anomalous transport exponent.

We remark that most of the applications reviewed in this section deal with asymptotic scaling only.

3 Weyl fractional differentiability and Hölder classes

The purpose of this section is to review the classical work of Stein, Zygmund and others [59, 60, 61, 62] which uses Weyl fractional calculus in the analysis of irregular functions. Let $f(x)$ be defined in a closed interval I, and let

$$\omega(\delta) = \omega(\delta : f) = \sup |f(x) - f(y)| \quad \text{for} \quad x, y \in I \quad |x - y| \leq \delta \tag{19}$$

The function $\omega(\delta)$ is called the modulus of continuity of f. If for some $\alpha > 0$ we have $\omega(\delta) \leq C\delta^\alpha$ with C independent of δ, then f is said to satisfy Hölder (In old literature it is called Lipschitz) condition of order α. These functions define a class of function Λ_α. Notice that with this definition of the modulus of continuity only the case $0 < \alpha \leq 1$ is interesting, since if $\alpha > 1$, $f'(x)$ is zero everywhere and the function is constant. Hence this definition has been generalized (see [28]) to one where the case $\alpha > 1$ is also nontrivial. The idea is to subtract an appropriate polynomial from a function f at y.

$$\tilde{\omega}(\delta) = \tilde{\omega}(\delta : f) = \sup |f(x) - P(x - y)| \quad \text{for} \quad x, y \in I \quad |x - y| \leq \delta \tag{20}$$

where P is the only polynomial of smallest degree which gives the smallest order of magnitude for $\tilde{\omega}$. Recently, this definition has been widely used [12, 13] to characterize the velocity field of a turbulent fluid.

Following Welland [62] we introduce

Definition 1 *f is said to have an αth derivative, where $0 \leq k < \alpha < k+1$ with integer k, if $D^{-\beta} f$, $\beta = k + 1 - \alpha$, has a $k + 1$ Peano derivatives at x_0 i.e. there exists a polynomial $P_{x_0}(t)$ of degree $\leq k + 1$ s. t.*

$$(D^{-\beta} f)(x_0 + t) - P_{x_0}(t) = o(|t|^{k+1}) \quad t \to 0. \tag{21}$$

Further if

$$\{\frac{1}{\rho} \int_{-\rho}^{\rho} |(D^{-\beta} f)(x_0 + t) - P_{x_0}(t)|^p dt\}^{1/p} = o(\rho^{k+1}) \quad \rho \to 0 \quad (1 \leq p < \infty) \tag{22}$$

f is said to have an αth derivative in the L^p sense. The $D^{-\beta}$ is in the Weyl sense.

It is clear that this definition of fractional differentiability is not local. Particularly the behavior of function at $-\infty$ also plays a crucial role. The main classical results can be stated using this notion of differentiability and involve the classes Λ^p_α and N^p_α which are given by the following definitons.

Definition 2 *If there exists a polynomial $Q_{x_0}(t)$ of degree $\leq k$ s. t. $f(x_0+t) - Q_{x_0}(t) = O(|t|^\alpha)$ as $t \to 0$ then f is said to satisfy the condition Λ_α and if*

$$\{\frac{1}{\rho}\int_{-\rho}^{\rho} |f(x_0+t) - Q_{x_0}(t)|^p dt\}^{1/p} = O(\rho^\alpha), \quad \rho \to 0 \quad (1 \leq p < \infty) \tag{23}$$

f is said to satisfy the condition Λ^p_α.

Definition 3 *f is said to satisfy the condition N^p_α if for some $\rho > 0$*

$$\frac{1}{\rho}\int_{-\rho}^{\rho} \frac{|f(x_0+t) - Q_{x_0}(t)|^p}{|t|^{1+p\alpha}} dt < \infty. \tag{24}$$

We are now in a position to state the classical results in the form of theorems 1 to 4. Next two theorems [61, 62] give the condition under which the fractional derivative of a function exists.

Theorem 1 *Suppose that f satisfies the condition Λ_α at every point of a set E of positive measure. Then $D^\alpha f(x)$ exists almost everywhere in E if and only if f satisfies condition N^2_α almost everywhere in E.*

Theorem 2 *The necessary and sufficient condition that f satisfies the condiiton N_α almost everywhere in a set E is that f satisfies the condition Λ^2_α and $D^\alpha f$ exisits in the L^2 sense, almost everywhere in this set.*

The following results in [59] tell us how the class of a function changes when an operation of fractional differentiation is performed.

Theorem 3 *Let $0 \leq \alpha < 1$, $\beta > 0$ and suppose that $f \in \Lambda_\alpha$. Then $D^{-\beta} f \in \Lambda_{\alpha+\beta}$ if $\alpha + \beta < 1$*

Theorem 4 *Let $0 < \gamma < \alpha < 1$,. Then $D^\gamma f \in \Lambda_{\alpha-\gamma}$ if $f \in \Lambda_\alpha$*

Though they have their own value, these results are not really adequate to obtain information regarding irregular behaviour of functions and Hölder exponents. We observe that the Weyl definition involves highly nonlocal information and hence is somewhat unsuitable for treatment of local scaling behaviour. In the next section we introduce a more appropriate definition.

4 Local Fractional Differentiability

In our previous work [36] we introduced the notion of local fractional derivative and demonstrated its use in the study of local scaling behaviour. We now briefly explain this notion and use it in subsequent sections.

4.1 Local fractional derivative and critical order

Recall the observations made in section 2, viz, (1) nonlocal information contained in fractional derivative and (2) the fractional derivative of a constant is not zero. The appropriate new notion of differentiability must bypass the hindrance due to these two properties. These difficulties can be remedied by introducing

Definition 4 *If, for a function $f : [0, 1] \to I\!R$, the limit*

$$I\!D^q f(y) = \lim_{x \to y} \frac{d^q(f(x) - f(y))}{d(x - y)^q} \tag{25}$$

exists and is finite, then we say that the local fractional derivative (LFD) of order q $(0 < q < 1)$, at $x = y$, exists.

In the above definition the lower limit y is treated as a constant. The subtraction of $f(y)$ corrects for the fact that the fractional derivative of a constant is not zero. Whereas the limit as $x \to y$ is taken to remove the nonlocal content. Advantage of defining local fractional derivative in this manner lies in its local nature and hence allowing the study of pointwise behaviour of functions. This will be seen more clearly in section 4.2 after the development of Taylor series.

Definition 5 *We define critical order α, at y, as*

$$\alpha(y) = Sup\{q \mid \text{all LFDs of order less than } q \text{ exist at } y\}.$$

Though these definitions are interesting only when the critical order is less than one, for the same reason as that for the first definition (section 3) of modulus of continuity we extend them for all values of $\alpha > 0$.

Definition 6 *If, for a function $f : [0, 1] \to I\!R$, the limit*

$$I\!D^q f(y) = \lim_{x \to y} \frac{d^q(f(x) - \sum_{n=0}^{N} \frac{f^{(n)}(y)}{\Gamma(n+1)}(x - y)^n)}{[d(x - y)]^q} \tag{26}$$

exists and is finite, where N is the largest integer for which N^{th} derivative of $f(x)$ at y exists and is finite, then we say that the local fractional derivative (LFD) of order q $(N < q < N + 1)$, at $x = y$, exists.

We consider this as the generalization of the local derivative for order greater than one. The definition of the critical order remains the same since, for $q < 1$, (25) and (26) agree. This definition extends the applicability of LFD to functions where the first derivative exists but are still irregular due to the nonexistence of some higher order derivative. As an example we note that the critical order of $f(x) = a + bx + c|x|^\gamma$ at origin, according to definitions 5 and 6 is γ.
Remark: 1)It is interesting to note that the same definition of LFD can be used for negative values of the critical order between -1 and 0. For this range of critical orders $N = -1$ and the sum in equation (26) is empty. As a result the expression for LFD becomes

$$I\!D^q f(y) = \lim_{x \to y} \frac{d^q f(x)}{[d(x - y)]^q} \tag{27}$$

An equivalence between the critical order and the Hölder exponent, for positive values of critical order, will be proved in section 6. Here we would like to point out that the negative Hölder exponents do arise in real physical situation of turbulant velocity field (see [63, 64] and references therein).
2)Another way of generalizing the LFD to the values of critical order beyond 1 would have been to write it as

$$\mathbb{D}^q f(y) = \lim_{x \to y} \frac{d^q (f^{(N)}(x) - f^{(N)}(y))}{[d(x-y)]^q} \tag{28}$$

But the existence of N^{th} derivative of f at x may not be guaranteed in general. Such a situation may arise in the case of multifractal functions to be treated in section 8.

4.2 LFD and generalized Taylor series

In order to see the information contained in the LFD we consider fractional Taylor's series with a remainder term for a real function f. Let

$$F(y, x-y; q, N) = \frac{d^q (f(x) - \sum_{n=0}^{N} \frac{f^{(n)}(y)}{\Gamma(n+1)} (x-y)^n)}{[d(x-y)]^q} \tag{29}$$

and

$$\widetilde{F_N}(x, y) = f(x) - \sum_{n=0}^{N} \frac{f^{(n)}(y)}{\Gamma(n+1)} (x-y)^n). \tag{30}$$

It is clear that

$$\mathbb{D}^q f(y) = F(y, 0; q, N) \tag{31}$$

Now, for $N < q < N+1$, it can be shown that

$$\begin{aligned} \widetilde{F_N}(x, y) &= \frac{\mathbb{D}^q f(y)}{\Gamma(q+1)} (x-y)^q \\ &+ \frac{1}{\Gamma(q+1)} \int_0^{x-y} \frac{dF(y, t; q, N)}{dt} (x-y-t)^q dt \end{aligned} \tag{32}$$

i.e.

$$f(x) = \sum_{n=0}^{N} \frac{f^{(n)}(y)}{\Gamma(n+1)} (x-y)^n + \frac{\mathbb{D}^q f(y)}{\Gamma(q+1)} (x-y)^q + R_1(x, y) \tag{33}$$

where $R_1(x, y)$ is a remainder given by

$$R_1(x, y) = \frac{1}{\Gamma(q+1)} \int_0^{x-y} \frac{dF(y, t; q, N)}{dt} (x-y-t)^q dt \tag{34}$$

We note that the local fractional derivative as defined above (not just fractional derivative), along with the first N derivatives, provides an approximation of $f(x)$ in the vicinity of y. We further note

that the terms on the RHS of eqn(32) are nontrivial and finite only in the case $q = \alpha$, the critical order. In ref.[65] a fractional Taylor series was constructed by Osler using usual (not local in the present sense) fractional derivatives. His results are, however, applicable to analytic functions and cannot be used for non-smooth functions directly. Further Osler's formulation involves terms with negative orders also and hence is not suitable for approximating schemes.

When $0 < q < 1$ we get as a special case

$$f(x) = f(y) + \frac{\mathbb{D}^q f(y)}{\Gamma(q+1)}(x-y)^q + Remainder \tag{35}$$

provided the RHS exists. One may note in equation(35) that when q is set equal to one in the above approximation one gets the equation of the tangent. It may be recalled that all the curves passing through a point y and having same tangent form an equivalence class (which is modelled by a linear behavior). Analogously all the functions (curves) with the same critical order α and the same $\mathbb{D}^\alpha$ will form an equivalence class modeled by power law x^α This is how one may generalize the geometric interpretation of derivatives in terms of tangents. This observation is useful when one wants to approximate an irregular function by a piecewise smooth (scaling) function.

5 Fractional Differentiability of Weierstrass Function

Consider a form of Weierstrass function, viz,

$$W_\lambda(t) = \sum_{k=1}^{\infty} \lambda^{(s-2)k} sin\lambda^k t, \quad \lambda > 1, \quad 1 < s < 2, \quad t \text{ real.} \tag{36}$$

Note that $W_\lambda(0) = 0$. The box dimension of the graph of this function is s. The Hausdorff dimension of its graph is still unknown. The best known bounds are given by Mauldin and Williams [66] where they have shown that there is a constant c such that

$$s - \frac{c}{\log \lambda} \leq dim_H graph f \leq s.$$

Recently it was shown [67] that if a random phase is added to each sine term in (36) then the Hausdorff dimension of the graph of the resulting function is s. These kind of functions have been studied in detail in [4, 68, 69, 70].

We note that there are dynamical systems with graphs of such functions as invariant sets. For example, let $g : \mathbb{R} \to \mathbb{R}$ be differentiable, and let $h : \mathbb{R}^2 \to \mathbb{R}^2$ be given by

$$h(x,t) = (\lambda t, \lambda^{2-s}(x-g)t)). \tag{37}$$

Then the graph of f can easily be seen to be a repeller of for h, where f is the function given by

$$f(t) = \sum_{k=1}^{\infty} \lambda^{(s-2)k} g(\lambda^k t). \tag{38}$$

In the following two subsections we prove lower and upper bound on critical order of the Weierstrass function.

5.1 Lower bound on critical order

To check the fractional differentiability at any point, say τ, we use $t' = t - \tau$ and $\widetilde{W}_\lambda(t',\tau) = W_\lambda(t'+\tau) - W_\lambda(\tau)$ so that $\widetilde{W}_\lambda(0,\tau) = 0$. We have

$$\begin{aligned}
\widetilde{W}_\lambda(t',\tau) &= \sum_{k=1}^{\infty} \lambda^{(s-2)k} sin\lambda^k(t'+\tau) - \sum_{k=1}^{\infty} \lambda^{(s-2)k} sin\lambda^k\tau \\
&= \sum_{k=1}^{\infty} \lambda^{(s-2)k}(cos\lambda^k\tau sin\lambda^k t' + sin\lambda^k\tau(cos\lambda^k t' - 1))
\end{aligned} \tag{39}$$

Now we take fractional derivative of this with respect to t'.

$$\frac{d^q\widetilde{W}_\lambda(t',\tau)}{dt'^q} = \sum_{k=1}^{\infty} \lambda^{(s-2+q)k}\left(cos\lambda^k\tau\frac{d^q sin\lambda^k t'}{d(\lambda^k t')^q} + sin\lambda^k\tau\frac{d^q(cos\lambda^k t' - 1)}{d(\lambda^k t')^q}\right) \tag{40}$$

From equations (6), (7) and second mean value theorem it follows that the fractional derivatives inside the above sum is bounded uniformly for all values of $\lambda^k t$. This implies that the series on the right hand side will converge uniformly for $q < 2-s$, justifying our action of taking the fractional derivative operator inside the sum.

Also as $t' \to 0$ for any k the fractional derivatives in the summation of equation (40) goes to zero. Therefore it is easy to see from this that

$$I\!\!D^q W_\lambda(\tau) = \lim_{t'\to 0}\frac{d^q\widetilde{W}_\lambda(t',\tau)}{dt'^q} = 0 \quad \text{for} \quad q < 2-s. \tag{41}$$

This shows that q^{th} local derivative of the Weierstrass function exists and is continuous, at any point, for $q < 2-s$.

5.2 Upper bound on critical order

For $q > 2-s$, right hand side of the equation (40) seems to diverge. We now prove that the LFD of order $q > 2-s$ in fact does not exist. To do this we write the Weierstrass function as follows.

$$W_\lambda(t) = \sum_{k=1}^{N} \lambda^{(s-2)k} sin\lambda^k t + \lambda^{(s-2)N} W_\lambda(\lambda^N t). \tag{42}$$

We now write

$$\begin{aligned}
\frac{d^q(W_\lambda(t) - W_\lambda(t'))}{d(t-t')^q} &= \sum_{k=1}^{N} \lambda^{(s-2+q)k}\frac{d^q(sin(\lambda^k t) - sin(\lambda^k t'))}{d(\lambda^k t)^q} \\
&\quad + \lambda^{(s-2+q)N}\frac{d^q(W_\lambda(\lambda^N t) - W_\lambda(\lambda^N t'))}{d(\lambda^N(t-t'))^q}
\end{aligned} \tag{43}$$

We choose N such that $\lambda^{-(N+1)} < |t-t'| \le \lambda^{-N}$. Now since $|sin(\lambda^k t) - sin(\lambda^k t')| \le \lambda^k|t-t'|$

$$\frac{d^q|sin(\lambda^k t) - sin(\lambda^k t')|}{d[\lambda^k(t-t')]^q} \le C\lambda^{k(1-q)}|t-t'|^{1-q} \tag{44}$$

This implies that the absolute value of the first term in equation (43) is bounded from above by $C\lambda^{(s-2+q)N}$. Now since the first derivative of the Weierstrass function does not exist at any point there exists a sequence of points t_n approaching t' such that $|W_\lambda(\lambda^N t_n) - W_\lambda(\lambda^N t')| \geq c\lambda^N |t_n - t'|$. Therefore there exists a sequence t_n such that

$$\frac{d^q |W(\lambda^N t_n) - W(\lambda^N t')|}{d[\lambda^N (t_n - t')]^q} \geq c\lambda^{N(1-q)} |t_n - t'|^{1-q} \tag{45}$$

and this is valid for every c for large enough n. This implies that

$$\lambda^{(s-2+q)N} \frac{d^q |W_\lambda(\lambda^N t_n) - W_\lambda(\lambda^N t')|}{d(\lambda^N (t_n - t'))^q} \geq c\lambda^{(s-2+q)N} \tag{46}$$

$$\begin{aligned} |\frac{d^q (W_\lambda(t_n) - W_\lambda(t'))}{d(t_n - t')^q} & - \lambda^{(s-2+q)N} \frac{d^q (W_\lambda(\lambda^N t_n) - W_\lambda(\lambda^N t'))}{d(\lambda^N (t_n - t'))^q}| \\ & \leq \sum_{k=1}^{N} \lambda^{(s-2+q)k} \frac{d^q |sin(\lambda^k t_n) - sin(\lambda^k t')|}{d(\lambda^k (t_n - t'))^q} \end{aligned} \tag{47}$$

Therefore we get

$$|\frac{d^q (W_\lambda(t_n) - W_\lambda(t'))}{d(t_n - t')^q} - c\lambda^{(s-2+q)N}| \leq C\lambda^{(s-2+q)k} \tag{48}$$

This implies that

$$|\frac{d^q (W_\lambda(t_n) - W_\lambda(t'))}{d(t_n - t')^q}| \geq C'\lambda^{(s-2+q)k} \tag{49}$$

for $C' > 0$. From this it is clear that the LFD of order greater that $2 - s$ does not exist. This concludes the proof.

Summarizing, therefore, the critical order of the Weierstrass function is $2 - s$ at all points. It may be noticed that this proof is valid for any $\lambda > 1$. This generalizes a similar result of [4, 36] which is valid only for sufficiently large λ. Thus there is a direct connection between dimension and the differentiability properties for $W_\lambda(t)$. As seen below, this observation is not restricted to $W_\lambda(t)$ but is quite general.

5.3 Lévy index of a Lévy flights and critical order

Schlesinger et al [71] have considered a Lévy flight on a one dimensional periodic lattice where a particle jumps from one lattice site to other with the probability given by

$$P(x) = \frac{\omega - 1}{2\omega} \sum_{j=0}^{\infty} \omega^{-j} [\delta(x, +b^j) + \delta(x, -b^j)] \tag{50}$$

where x is magnitude of the jump, b is a lattice spacing and $b > \omega > 1$. $\delta(x, y)$ is a Kronecker delta. The characteristic function for $P(x)$ is given by

$$\tilde{P}(k) = \frac{\omega - 1}{2\omega} \sum_{j=0}^{\infty} \omega^{-j} cos(b^j k). \tag{51}$$

which is nothing but the Weierstrass function. For this distribution the Lévy index is $\log\omega/\log b$, which can be identified as critical order of $\tilde{P}(k)$.

More generally for the Lévy distribution with index μ the characteristic function is given by

$$\tilde{P}(k) = A\exp c|k|^{\mu}. \tag{52}$$

Critical order of this function at $k = 0$ also turns out to be same as μ. Thus the Lévy index can be identified as the critical order of the characteristic function at $k = 0$.

6 Equivalence between critical order and the Hölder exponent

We recall the definition of Hölder exponent $h(y)$ of a function f at y [12, 13] as h such that there exists a polynomial $P_n(x)$ of order n, where n is the largest integer smaller than h, that satisfies

$$|f(x) - P_n(x - y)| = O(|x - y|^h), \tag{53}$$

for x in the neighbourhood of y. When h is restricted between 0 and 1 the equation (53) takes the form

$$|f(x) - f(y)| = O(|x - y|^h), \tag{54}$$

The Hölder exponent characterizes the behaviour of the function around a given point. One can study the function at every point and find its pointwise Hölder exponents. According to equation (53) an analytic function has $h = \infty$ at every point.

As can be seen clearly that the definition of the Hölder exponent is not algorithmic and hence methods need to be developed for its determination. In [36] the following result was proved in a slightly different form, which establishes an equivalence between the Hölder exponent and the critical order.

Theorem 5 *The continuous function $f(x)$ has Hölder exponent α, $0 < \alpha < 1$, at y iff $I\!\!D^q f(y) = 0$ for all $q < \alpha$ and $I\!\!D^q f(y)$ does not exist for $1 > q > \alpha$.*

This result can be understood from the simple example of fractional derivative of a power x^p, given in equation (5). In this equation if $0 < p < 1$ and $0 < q < 1$ then the RHS exists. But if we take a limit $x \to 0$ then we get

$$\lim_{x\to 0} \frac{d^q x^p}{dx^q} = \begin{cases} 0 & \text{if } q < p \\ \text{const} & \text{if } p = q \\ \infty & \text{otherwise} \end{cases} \tag{55}$$

The LHS in the above equation is nothing but the LFD of x^p. This shows that the critical order gives the exponent p. This is expected to happen for any function and at any point with a local power law behaviour.

The following corrollary follows from the theorem 5 and a well known result giving relation between Hölder exponent and box dimension of a graph of a fractal function [4].

Corollary 1 *If the critical order of a function $f(x)$ at every point x is α then $dim_B f = 2 - \alpha$ where $dim_B f$ is a box dimension of the graph of the function f.*

With a slight modification in the proof of theorem 5 a general result giving equivalence between the Hölder exponent and the critical order using the general definition of LFD follows. The functions F and $\widetilde{F}$ are defined in equations (29) and (30) respectively.

Theorem 6 *Let $f : [0,1] \to I\!R$ be a continuous function.*
a)If $I\!D^q f(y) = 0$ where $N < q < N+1$, for some y, then $h(y) \geq q$.
b)If there exists a sequence $x_n \to y$ as $n \to \infty$ such that

$$\lim_{n\to\infty} F(y, x_n - y; q, N) = \pm\infty \quad ,$$

for some y, then $h(y) \leq q$.

Similarly a following converse of the above theorem can also be proved.

Theorem 7 *Let $f : [0,1] \to I\!R$ be a continuous function.*
a) Suppose

$$|\widetilde{F}_N(x,y)| \leq c|x-y|^\alpha,$$

where $c > 0$, $N < \alpha < N+1$ and $|x-y| < \delta$ for some $\delta > 0$. Then $I\!D^q f(y) = 0$ for any $q < \alpha$ for $y \in (0,1)$
b) Suppose that for $y \in (0,1)$ and for each $\delta > 0$ there exists x such that $|x-y| \leq \delta$ and

$$|\widetilde{F}_N(x,y)| \geq c\delta^\alpha,$$

where $c > 0$, $\delta \leq \delta_0$ for some $\delta_0 > 0$ and $0 < \alpha < 1$. Then there exists a sequence $x_n \to y$ as $n \to \infty$ such that

$$\lim_{n\to\infty} F(y, x_n - y; q, N) = \pm\infty \quad \text{for} \quad q > \alpha$$

These two theorems give an equivalence between Hölder exponent and the critical order of fractional differentiablity. Their proofs are similar to that of theorem 5.

7 Isolated masked singularities

The purpose of this section is to demonstrate the use of LFD to detect masked singularities. We will consider only isolated singularities. We choose the simplest example $f(x) = \sum_{n=0}^{N} a_n x^n + a x^\alpha, \quad N < \alpha < N+1, \quad x > 0$. Critical order at $x = 0$ gives the order of singularity at that point whereas the value of the LFD $I\!D^{q=\alpha} f(0)$, viz $a\Gamma(\alpha+1)$, gives strength of the singularity.

Using LFD we can detect a weaker singularity masked by a stronger singularity. As demonstrated below, we can estimate and subtract the contribution due to stronger singularity from the function and find out the critical order of the remaining function. Consider, for example, the function

$$f(x) = \sum_{n=0}^{N} a_n x^n + a x^\alpha + \sum_{n=N+1}^{M} a_n x^n + b x^\beta, \tag{56}$$

where $N < \alpha < N+1 < M < \beta < M+1$ and $x > 0$. LFD of this function at $x = 0$ of the order α is $I\!D^{\alpha} f(0) = a\Gamma(\alpha+1)$. Using this estimate of stronger singularity we now write

$$G(x;\alpha) = f(x) - \sum_{n=0}^{N} \frac{f^{(n)}(0)}{\Gamma(n+1)} x^n - \frac{I\!D^{\alpha} f(0)}{\Gamma(\alpha+1)} x^{\alpha}.$$

The critical order of this function, at $x = 0$, is β which is a masked singularity. Notice that the estimation of the weaker singularity was possible in the above calculation just because the LFD gave the coefficient of $x^{\alpha}/\Gamma(\alpha+1)$. This suggests that using LFD, one should be able to extract secondary singularity spectrum masked by the primary singularity spectrum of strong singularities. Hence one can gain more insight into the processes giving rise to irregular behavior.

Comparison of two methods of studying pointwise behavior of functions, one using wavelets and the other using LFD, shows that characterisation of Hölder classes of functions using LFD is direct and involves fewer assumptions. Characterisation of Hölder class of functions with oscillating singularity, e.g. $f(x) = x^{\alpha} sin(1/x^{\beta})$ ($x > 0$, $0 < \alpha < 1$ and $\beta > 0$), using wavelets needs two exponents [27]. Using LFD, owing to theorem I and II critical order directly gives the Hölder exponent for such a function.

It has been shown in the context of wavelet transforms that one can detect singularities masked by regular polynomial behavior [12] by choosing a appropriate analysing wavelet. (Wavelets with first n, for some suitable n, moments vanishing are considered appropriate). If one has to extend the wavelet method to unmask weaker singularities, one would then require analysing wavelets with fractional moments vanishing. Notice that one may require this condition along with the condition on first n moments. Further the class of functions to be analysed is in general restricted in these analyses. These restrictions essentially arise from the asymptotic properties of the wavelets used. On the other hand, these restrictions are not relevant while using LFD.

8 Multifractal function

We saw in section 5 that the Weierstrass function is a fractal function, i.e., it has the same Hölder exponent at every point. But there are multifractal functions which have different Hölder exponents at different points. These functions can be used to model various intermitent signals arising in physical applications. Since the critical order gives the local and pointwise behavior of the function, conclusions of the theorem 5, 6 and 7 will carry over even to the case of multifractal functions where we have different Hölder exponents at different points. Selfsimilar multifractal functions have been constructed by Jaffard [13]. We give one specific example of such a function. This function is a solution F of the functional equation

$$F(x) = \sum_{i=1}^{d} \lambda_i F(S_i^{-1}(x)) + g(x), \quad x \text{ real.} \tag{57}$$

where S_i's are the affine transformations of the kind $S_i(x) = \mu_i x + b_i$ (with $|\mu_i| < 1$ and b_i's real) and λ_i's are some real numbers and g is any sufficiently smooth function (it is assumed that g and its derivatives have fast decay). For the sake of illustration we choose $\mu_1 = \mu_2 = 1/3$, $b_1 = 0$, $b_2 = 2/3$, $\lambda_1 = 3^{-\alpha}$, $\lambda_2 = 3^{-\beta}$ ($0 < \alpha < \beta < 1$) and

$$\begin{aligned} g(x) &= sin(2\pi x) \qquad \text{if} \quad x \in [0,1] \\ &= 0 \qquad \text{otherwise.} \end{aligned}$$

Such functions are studied in detail in [13] using wavelet transforms where it was shown that the above functional equation (with the parameters we have chosen) has a unique solution F. Further at any point F either has Hölder exponents ranging from α to β or is smooth. A sequence of points $S_{i_1}(0),\ \ S_{i_2}S_{i_1}(0),\ \ \cdots,\ \ S_{i_n}\cdots S_{i_1}(0),\ \ \cdots$, where i_k takes values 1 or 2, tends to a point in $[0,1]$ (in fact to a point of a triadic Cantor set) and for the values of μ_is we have chosen this correspondence between sequences and limits is one to one. The solution of the above functional equation is given by [13]

$$F(x) = \sum_{n=0}^{\infty} \sum_{i_1,\cdots,i_n=1}^{2} \lambda_{i_1}\cdots\lambda_{i_n} g(S_{i_n}^{-1}\cdots S_{i_1}^{-1}(x)). \tag{58}$$

Note that with the above choice of parameters the inner sum in (58) reduces to a single term. Jaffard [13] has shown that the local Hölder exponent at y is

$$h(y) = \liminf_{n\to\infty} \frac{\log\left(\lambda_{i_1(y)}\cdots\lambda_{i_n(y)}\right)}{\log\left(\mu_{i_1(y)}\cdots\mu_{i_n(y)}\right)}, \tag{59}$$

where $\{i_1(y)\cdots i_n(y)\}$ is a sequence of intergers appearing in the sum in equation(58) at a point y. It is clear that $h_{min} = \alpha$ and $h_{max} = \beta$. The function F at the points of a triadic cantor set have $h(x) \in [\alpha, \beta]$ and at other points it is smooth (where F is as smooth as g). Benzi et. al. [72] have constructed multifractal functions which are random in nature unlike the above nonrandom functions. For still another approach also see [73].

Several well-known 'pathological' functions have been reanalyzed in [29, 30, 31, 32] and found to have multifractal nature. Here we consider one example of classical multifractal function.

$$R(x) = \sum_{n=1}^{\infty} \frac{1}{n^2}\sin(\pi n^2 x) \tag{60}$$

This function was proposed by Riemann. It turns out that the regularity of this function varies strongly from point to point. Hardy and Littlewood [74] proved that $R(x)$ is not differentiable at x_0 if x_0 is irrational or if x_0 can not be written as $2p+1/2q+1$ $(p, q \in N)$. In fact they showed that the Hölder exponent at these points $\leq 3/4$. Gerver [75] proved the differentiability of $R(x)$ at points of the form $2p+1/2q+1$ $(p, q \in N)$. At these points the Hölder exponent is $3/2$. This function has also been studied in [76, 77]. Jaffard [29] has recently shown that the dimension spectrum of the Riemann function is given as below.

$$d(\alpha) = \begin{cases} 4\alpha - 2 & \text{if} \quad \alpha \in [\frac{1}{2}, \frac{3}{4}] \\ 0 & \text{if} \quad \alpha = \frac{3}{2} \\ -\infty & \text{otherwise} \end{cases} \tag{61}$$

where $d(\alpha)$ gives the Hausdorff dimension of the set when the Hölder exponent is α.

LFD forms one method of studying pointwise behaviour of such multifractal functions alongwith other methods and may cosiderably reduce the analysis involved. This fact was demonstrated in [36] on a specific example of self-similar multifractal function given by equation (58).

9 Concluding remarks

First we reviewed the applications of various fractional differential equations in different physical situations. It was noted that most of these applications dealt with the asymptotic scaling. Further we reviewed the classical results within the framework of Weyl fractional calculus. However these results were found to be inadequate for the study of pointwise behavior of fractal and multifracal functions. The notion of LFD as developed in [36] was found suitable for this purpose. In terms of the LFD it was also possible to write a fractional Taylor series of a function (useful in an analytic treatment of approximations). It was pointed out that generalization of the notion of tangents to the graph of a function (useful for geometric purposes) is also possible using LFD.

It was established that the critical order of the Weierstrass function is related to the box dimension of its graph. If the dimension of the graph of such a function is $1+\gamma$, the critical order is $1-\gamma$. When γ approaches unity the function becomes increasingly irregular and local fractional differentiability is lost accordingly. Thus there is a direct quantitative connection between the dimension of the graph and the fractional differentiability property of the function. This is one of the remarkable conclusions of the new approach. An important consequence of this approach is that a classification of continuous paths (e.g. fractional Brownian paths) or functions according to local fractional differentiablity properties is also a classification according to dimensions (or Hölder exponents).

Also the Lévy index of a Lévy flight on one dimensional lattice is identified as the critical order of the characteristic function of the walk. More generally, the Lévy index of a Lévy distribution is identified as the critical order of its characteristic function at the origin.

We have argued and demostrated that LFDs are useful for studying isolated singularities and singularities masked by the stronger singularity (not just by regular behavior). It was also shown that the pointwise behavior of irregular (fractal or multifractal) functions can be studied using the methods of this paper.

We note, however, that the treatment of random irregular functions as well as multivariable irregular functions is badly needed. We hope that these problems can be tackled in near future.

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References

[1] T. W. Körner *Fourier Analysis* (Cambridge University Press, Cambridge, 1989)

[2] For historical remarks and construction see, for example, [1, 3, 4].

[3] B. B. Mandelbrot *The Fractal Geometry of Nature* (Freeman, New York, 1977).

[4] K. Falconer *Fractal Geometry* (John Wiley, New York, 1990).

[5] R. P. Feynmann and A. R. Hibbs *Quantum Mechanics and Path Integrals* (McGraw-Hill, New York, 1965).

[6] L. F. Abott and M. B. Wise *Am. J. Phys.* **49**, 37 (1981).

[7] P. Constantin, I. Procaccia and K. R. Sreenivasan *Phys. Rev. Lett.* **67**, 1739 (1991).

[8] P. Constantin and I. Procaccia *Nonlinearity* **7**, 1045 (1994).

[9] K. Sarkar and C. Meneveau *Phys. Rev. E* **47**, 957 (1993).

[10] J. L. Kaplan, J. Malet-Peret and J. A. Yorke *Ergodic Th. and Dyn. Syst.***4**, 261 (1984).

[11] J. D. Farmer, E. Ott and J. A . Yorke Physica D **7**, 153 (1983).

[12] J. F. Muzy, E. Bacry and A. Arneodo *Phys. Rev. E* **47**, 875 (1993).

[13] S. Jaffard, To appear in *SIAM J. of Math. Anal.*.

[14] U. Frisch and G. Parisi, in *Turbulence and Predictability in Geophysical Fluid Dynamics and Climate Dynamics* edited by M Ghil, R. Benzi and G. Parisi (North-Holland, Amsterdam, 1985).

[15] R. Benzi, G. Paladin, G. Parisi and A. Vulpiani *J. Phys. A* **17**, 3521 (1984).

[16] T. C. Halsey, M. H. Jensen, L. P. Kadanoff, I. Procaccia and B. I. Shraiman *Phys. Rev. A* **33**, 1141 (1986).

[17] P. Collet, J. Lobowitz and A. Porzio *J. Stat. Phys.* **47**, 609 (1987).

[18] M. H. Jensen, L. P. Kadanoff and I. Procaccia *Phys. Rev. A* **46**, 1409 (1987).

[19] B. B. Mandelbrot *Pure Appl. Geophys.* **131**, 5 (1989).

[20] C. Meneveau and K. R. Sreenivasan, *Phys. Rev. Lett.* **59**, 1424 (1987).

[21] A. B. Chhabra and K. R. Srinivasan, *Phys. Rev. Lett.* **68** 2762 (1992).

[22] J. Feder, *Fractals* (Pergamon, New York, 1988).

[23] T. Vicsek, *Fractal Growth Phenomenon* (World Scientific, Singapore, 1989).

[24] R Cawley and R. D. Mauldin, *Adv. in Maths* **92**, 192 (1992).

[25] M. Holschneider *J. Stat. Phys.* **77**, 807 (1994).

[26] S. Jaffard, in *Wavelets and Applications* ed. Meyer Y. (Springer-Verlag, Berlin, 1992).

[27] A. Arneodo, E. Bacry and J. F. Muzy *Phys. Rev. Lett.* **74**, 4823 (1995).

[28] S. Jaffard and Y. Meyer, to appear in *Memoirs of the AMS*.

[29] S. Jaffard *Preprint*

[30] S. Jaffard and B. B. Mandelbrot, to appear in *Advances in Maths.*

[31] I. Daubechies and J. Lagarias, *Rev. of Math. Phys.* **6**, 1033 (1994).

[32] S. Jaffard, *Preprint*

[33] K. B. Oldham and J. Spanier *The Fractional Calculus* (Academic Press, New York, 1974)

[34] K. S. Miller and B. Ross *An Introduction to the Fractional Calculus and Fractional Differential Equations* (John Wiley, New York, 1993)

[35] B. Ross, in *Fractional Calculus and its Applications: Lecture Notes in Mathematics* (Springer, New York, 1975) vol. 457 p. 1.

[36] K. M. Kolwankar and A. D. Gangal *Chaos* **6**, 505 (1996). (chao-dyn/9609016)

[37] R. Hilfer *Phys. Scr.* **44**, 321 (1991).

[38] R. Hilfer *Phys. Rev. Lett.* **68**, 190 (1992).

[39] K. M. Kolwankar, *Preprint*

[40] T. F. Nonnenmacher *J. Phys. A: Math. Gen.* **23**, L697 (1990)

[41] M. Giona and H. E. Roman *J. Phys. A: Math Gen.* **25**, 2093 (1992)

[42] H. E. Roman and M. Giona *J. Phys. A: Math. Gen.* **25**, 2107 (1992)

[43] N. Patzschke and M. Zähle *Stochastic Process Appl.* **43**, 165 (1992)

[44] B. B. Mandelbrot and J. W. Van Ness *SIAM Rev.* **10**, 422 (1968).

[45] W. G. Glöckle and T. F. Nonnenmacher *J. Stat. Phys.* **71**, 741 (1993).

[46] M. F. Schlesinger *J. Stat. Phys.* **36**, 639 (1984).

[47] K. L. Sebastian *J. phys. A: Math. Gen.* **28**, 4305 (1995).

[48] J. P. Bouchaud and A. Georges *Phys. Rep.* **195**, 127 (1990).

[49] M. F. Shlesinger, B. J. West and J. Klafter *Phys. Rev. Lett.* **58**, 1100 (1987).

[50] M. F. Shlesinger, G. M. Zaslavsky and J. Klafter *Nature* (London) **363**, 31 (1993).

[51] M. F. Shlesinger et. al., Eds, *Lévy Flights and Related Topics in Physics* (Springer, Berlin, 1995).

[52] T. H. Solomon, E. R. Weeks and H. L. Swinney *Phys. Rev. Lett.* **71**, 3975 (1993).

[53] A. Ott, J. P. Bouchaud, D. Langvin and W. Urbach *Phys. Rev. Lett.* **65**, 2201 (1990).

[54] W. Wyss J. Math. Phys. **27**, 2782 (1986).

[55] W. R. Schneider and W. Wyss *J. Math. Phys.* **30**, 134 (1989).

[56] G. Jumarie *J. Math. Phys.* **33**, 3536 (1992).

[57] H. C. Fogedby *Phys. Rev. Lett.***73**, 2517 (1994).

[58] G. M. Zaslavsky *Physica D* **76**, 110 (1994).

[59] A. Zygmund *Trignometric series* Vols I, II, 2nd ed., Cambridge Univ. Press, New York, 1959.

[60] E. M. Stein *Singular integrals and differentiability properties of functions* (Princeton University press, Princeton, 1970)

[61] E. M. Stein and A. Zygmund *Proc. London Math. Soc.* (3) **14a** 249 (1965).

[62] G. V. Welland *Proc. Amer. Math. Soc.* **19**, 135(1968).

[63] I. Eyink *J. Stat. Phys.* **78**, 353 (1995).

[64] S. Jaffard, To appear in *SIAM J. of Math. Anal.*.

[65] T. J. Osler *SIAM J. Math. Anal.* **2**, 37 (1971).

[66] R. D. Mauldin and S. C. Williams, *Trans. Am. Math. Soc.* **298** (2), 793 (1986).

[67] B. R. Hunt, Preprint (1996).

[68] G. H. Hardy *Tr. Am. Math. Soc.* **17**, 301 (1916).

[69] A. S. Besicovitch and H. D. Ursell *J. Lond. Math. Soc.* **12**, 18 (1937).

[70] M. V. Berry and Z. V. Luwis *Proc. R. Soc.***A370**, 459 (1980).

[71] M. F. Shlesinger *Physica D* **38**, 304 (1989).

[72] R. Benzi et. al. *Physica D*, **65**, 352 (1993).

[73] Juneja et. al., *Phys. Rev. E* **41** 5179 (1994).

[74] G. H. Hardy and E. Littlewood *Acta Math.* **37**, 194 (1914).

[75] J. Gerver *Am. J. Math.* **93**, 33 (1970).

[76] M. Holschneider and Ph. Tchamitchian *Invent. Math.* **105**, 157 (1991).

[77] J. J. Duistermaat, *Overdruk* **9**, 303 (1991).

[78] L. Olsen, *Pitman Research Notes in Mathematica Series* **307** (1994).

[79] R. Riedi, *J. Math. Anal. Appl.* **189** 462 (1995).

[80] M. Zähle, Friedrich-Schiller Universität Jena, Preprint (1996).

[81] M. Zähle, *Fractals* **3** 747 (1995).